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Torsional oscillations of a plane lamina in a buoyant fluid

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Abstract. The correction to the torque on a torsionally oscillating plane in a viscous, thermally conducting fluid due to its density variations is calculated when the thickness of the shear layers is small in comparison to the dimensions of the plane. The mechanism that drives the system away from strictly isothermal conditions is viscous dissipation. The corresponding variations in the mean (axial and radial) flow and the mean temperature are investigated for relatively strong and weak buoyancy-momentum coupling. Method of matched asymptotic expansions is employed to study the resulting fluid motion which exists as a double-decker structure and depends upon three physical parameters.

1. Introduction

Carrier and DiPrima [1], who investigated the torsional oscillations of a solid sphere in an unbounded viscous fluid, pointed out that the torsional oscillations induce a secondary flow in planes containing the axis of rotation. An algebraic error in their calculations was subsequently corrected by DiPrima and Liron [2]. Rosenblat [3], in his analysis of the torsional oscillations of a plane lamina, established that not only is a secondary flow induced, but in addition to the Stokes layer there is a second thicker viscous boundary layer associated with the secondary flow. Benney [4], later on, confirmed this. Chawla and Verma [5] proved that the outer viscous layer is thicker than the one predicted by Rosenblat [3] and Benney [4]. The existance of double boundary layers in oscillatory viscous flow in general has been discussed by Stuart [6] and Riley [7]. Folse, Hussey and Bosnak [8] considered finite-amplitude effects in a series of measurements of the periodic and logarithmic decrement of a torsionally oscillating disk. Their measurements are in qualitative agreement with the theoretical calculations derived from Rosenblat's results. Schippers [9], on the other hand, obtained numerical solutions, spanning uniformly whole of the flow regime, for specific values of frequency parameter. The specific purpose of this paper is to study how these characteristics are modified when the plane oscillates in contact with a thermally conducting viscous fluid as affected by the buoyancy force.

The viscous dissipation of the fluid in the neighbourhood of the bounding plane heats up the fluid, imparting density variations responsible for meridional circulation. This circulation balances the outward thermal and viscous diffusion which results in the establishment of a double-decker layer of the viscous flow. The heating up of the fluid also decreases the torque of the fluid experienced by the oscillating plane. The thickness of the outer boundary layers associated with axial and radial flow increases as the buoyancy forces weaken in comparison to the viscous force.

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The existence of two or more regions of flow necessitates the use of a suitable technique to avoid the complications of a singular expansion. To this end, the direct extension of the von Kármán radial similarity to the thermal field alongwith the method of matched asymptotic expansions enables us to develope an axisymmetric solution of the governing equations. The procedure, adopted in the present paper, provides a direct approach to an identification of the effect of the buoyancy-momentum coupling, leading thereby to the proper significance of each of the physical parameters.

2. Mathematical formulation

We take the plane lamina to be isothermal, of infinite extent, and coincident with the plane $Z = 0$. The region $Z \geqslant 0$, above the plane, is occupied by a viscous incompressible fluid. The plane is kept at the same temperature (T_∞) as the ambient temperature. The plane lamina, in contact with the fluid, is performing torsional oscillations of angular frequency ω and amplitude Ω about the Z-axis which is aligned with the negative direction of gravity g. We choose cylindrical coordinates (r, θ, Z) with τ as time, and denote by V, p, and T the fluid velocity, the hydrodynamic pressure and the fluid temperature respectively. Moreover, ρ, ν, k, c and β are respectively the density, the kinematic viscosity, the thermal diffusivity, specific heat and the coefficient of thermal expansion of the fluid.

Consistent with axial symmetry and the continuity equation, we introduce similarity functions $H(z,t), G(z,t), P_0(z,t), P(z,t), S(z,t)$ and $Q(z,t)$ and write

$$V = \Omega[-\frac{1}{2}rH_z\hat{r} + rG\hat{\theta} + (\nu/w)^{1/2}H\hat{z}], \tag{2.1}$$

$$p = -\rho\nu\Omega P_0 + \rho r^2\omega^2 P, \tag{2.2}$$

$$T = T_\infty + (r^2\Omega^2/c)S + (\nu\Omega/c)Q \tag{2.3}$$

where z and t are dimensionless variables defined by

$$z = (\omega/\nu)^{1/2}Z, \quad t = \omega\tau, \tag{2.4}$$

and $\hat{r}, \hat{\theta}$ and $\hat{z}$ are unit vectors in the r-, θ- and z-directions respectively. It has been shown by Rotem and Classen [10] that the existence of the similarity solution is limited to the temperature distribution of the form (3). Substituting (1) to (4) into the basic momentum and energy equations (with the dissipation terms retained), we obtain

$$H_{zzz} - H_{zt} = \varepsilon[HH_{zz} - \frac{1}{2}H_z^2 + 2G^2 - 4P], \tag{2.5}$$

$$G_{zz} - G_t = \varepsilon[HG_z - GH_z], \tag{2.6}$$

$$P_z = \varepsilon\gamma S, \tag{2.7}$$

$$H_{zz} - H_t = P_{oz} + \varepsilon[HH_z - \gamma Q], \tag{2.7a}$$

$$\sigma^{-1}S_{zz} - S_t + G_z^2 + \frac{1}{4}H_{zz}^2 = \varepsilon[HS_z - SH_z], \tag{2.8}$$

$$\sigma^{-1}Q_{zz} - Q_t = \varepsilon[HQ_z - 4\sigma^{-1}S - 3H_z^2], \tag{2.9}$$

where ε, γ and σ are dimensionless parameters defined by

$$\varepsilon = \Omega/\omega, \quad \gamma = g\beta(\nu\omega)^{1/2}/c\Omega, \quad \sigma = \nu/k. \tag{2.10}$$

Here σ is the Prandtl number of the fluid and γ (a modified Grashof number) measures the relative strength of the buoyancy force. The variation in density is taken into account only in the derivation of the buoyancy force (appearing only in the Z-component of the momentum equation) while other variations are neglected within the framework of the constant property fluid. The appropriate boundary conditions are:

$$H = 0 = H_z, \quad G = \cos t, \quad S = 0 = Q \quad \text{at} \quad z = 0, \tag{2.11}$$

$$H_z \to 0, \quad G \to 0, \quad S \to 0, \quad Q \to 0, \quad P \to 0 \quad \text{as} \quad z \to \infty. \tag{2.12}$$

Since the fluid motion induced by the torsional vibrations of the plane lamina is not affected by equation (9), its solution is not being sought. Moreover, equation (7a) yields the vertical pressure gradient part P_{oz}. We, therefore concern ourselves with the solution of the coupled system (5), (6), (7) and (8) for $\varepsilon \ll 1$.

In order to expose the effect of the buoyancy force on the induced fluid motion over the whole of the flow regime, we employ the method of matched asymptotic expansions. The method is effective largely because the entire physical system has a distinct multi-layered structure. The unsteady inner layers are the well known Stokes layers which result from the skin-effect. Under the buoyancy-diffusion-convection interaction, the thickness of the outer region depends critically on the orders of magnitude of γ and σ. The orders of magnitude of γ considered in this paper are $\gamma = o(1)$ and $\gamma = o(\varepsilon^2)$ with $\sigma = o(1)$.

3. Solution with $\gamma = o(1), \sigma = o(1)$

For $\varepsilon < 1$, it is appropriate to develop each of the unknown functions H, G, P, and S as asymptotic series in ε. Such a solution will, however, be valid for the inner layers only. The boundary conditions on G and the location and composition of different terms in the governing equations (5) to (8) suggest that the azimuthal velocity field G consists of only odd harmonics in time, whereas the radial and axial flow field, given by H, and the thermal field component S contain even harmonics together with steady terms. But G must be uniform in the far field as is evident from (6). Hence $G = 0$ in the outer region (see also (19)). Both H and S penetrate outside the innder layers. It is clear from equation (8) that the penetration of S in the outer region is of $o(1)$. Taking $z = o(\varepsilon^i), H = o(\varepsilon^m)$ and $P = o(\varepsilon^l)$ in equations (5) and (6), a simple order-of-magnitude analysis reveals that, for $\gamma = o(1)$, the outer layers are $o(\varepsilon^{-3/5})$ times as thick as the inner layers, with $m = -2/5$ and $l = 2/5$. In the outer region we, therefore, set

$$\begin{aligned} \eta &= \varepsilon^{3/5} z, \quad H(z,t) = \varepsilon^{-2/5} h(\eta, t), \quad S(z,t) = s(\eta, t), \\ P(z,t) &= \varepsilon^{2/5} p(\eta, t), \quad G(z,t) = 0, \end{aligned} \tag{3.13}$$

so that the functions h, p and s are given by

$$h_{\eta t} = \varepsilon^{6/5}[h_{\eta\eta\eta} - h h_{\eta\eta} + \frac{1}{2} h_\eta^2 - 4p], \tag{3.14}$$

$$s_t = \varepsilon^{6/5}[\sigma^{-1} s_{\eta\eta} - h s_\eta + s h_\eta] + \frac{1}{4}\varepsilon^{8/5} h_{\eta\eta}^2, \tag{3.15}$$

$$p_\eta = \gamma s. \tag{3.16}$$

Writing the inner solution as $H(z,t) = \sum_{n=0}^{\infty} \varepsilon^{n/5} H_n(z,t)$ etc., and the outer solution as $h(z,t) = \sum_{n=0}^{\infty} \varepsilon^{n/5} h_n(\eta,t)$ etc., and employing the usual matching conditions, namely $H(\infty,t) = \varepsilon^{-2/5} h(0,t)$ etc., we arrive at the following solution valid for the Stokes (inner) layers:

$$\begin{aligned} S(z,t) = {} & \frac{\sigma}{4}(1 - \mathrm{e}^{-\sqrt{2}z}) + \frac{\sigma}{4(\sigma-2)}[\mathrm{e}^{-\sqrt{2}z}\cos(2t - \sqrt{2}z) \\ & - \mathrm{e}^{-\sigma^{1/2}z}\cos(2t - \sqrt{\sigma}z)] + \varepsilon^{3/5} b_3 z + \varepsilon^{7/5} b_7 z + o(\varepsilon^{9/5}), \end{aligned} \tag{3.17}$$

$$\begin{aligned} H(z,t) = {} & \varepsilon^{4/5} a_4 z^2 + \varepsilon[\frac{1}{2\sqrt{2}} - \frac{z}{2} - \frac{1}{2\sqrt{2}}\mathrm{e}^{-\sqrt{2}z} \\ & + \frac{1}{4}\{\mathrm{e}^{-\sqrt{2}z}\cos(2t - \sqrt{2}z + \frac{\pi}{4}) - \sqrt{2}\mathrm{e}^{-z}\cos(2t - z + \frac{\pi}{4}) \\ & + (\sqrt{2}-1)\cos(2t + \frac{\pi}{4})\}] + \varepsilon^{7/5} a_7 z^3 + \varepsilon^{8/5} a_8 z^2 \\ & + \varepsilon^2[\sigma\gamma(\frac{1}{4}\mathrm{e}^{-\sqrt{2}z} - \frac{1}{4} + \frac{z}{2\sqrt{2}} - \frac{z^4}{24}) + \frac{\gamma}{4(\sigma-1)(\sigma-2)} \\ & \times \{\sigma^{1/2}\mathrm{e}^{-z}\cos(2t - z) - \mathrm{e}^{-\sigma^{1/2}z}\cos(2t - \sqrt{\sigma}z)\} \\ & - \frac{\gamma\sigma}{8(\sigma-2)}\{\sqrt{2}\mathrm{e}^{-z}\cos(2t - z) - \mathrm{e}^{-\sqrt{2}z}\cos(2t - \sqrt{2}z)\} \\ & + \frac{\gamma\{(\sqrt{2}-1)\sigma + \sigma^{1/2} + \sqrt{2}\}}{8(\sigma^{1/2}+1)(\sigma^{1/2}+1)(\sigma^{1/2}+\sqrt{2})}\cos 2t] + o(\varepsilon^{13/5}), \end{aligned} \tag{3.18}$$

$$\begin{aligned} G(z,t) = {} & \mathrm{e}^{-z/\sqrt{2}}\cos(t - \frac{z}{\sqrt{2}}) + \varepsilon^{9/5} a_4[\frac{z^2}{6}\cos(t - \frac{z}{\sqrt{2}}) \\ & + \frac{3z^2}{4}\cos(t - \frac{z}{\sqrt{2}} - \frac{\pi}{4}) + \frac{3z}{4}\cos(t - \frac{z}{\sqrt{2}} - \frac{\pi}{2})]\,\mathrm{e}^{-z/\sqrt{2}} \\ & + \varepsilon^2[\frac{1}{48}(\mathrm{e}^{-3z/\sqrt{2}}\cos(3t - \frac{3z}{\sqrt{2}}) - \mathrm{e}^{-\sqrt{\frac{3}{2}}z}\cos(3t - \sqrt{\frac{3}{2}}z) \\ & + 3(1-\sqrt{2})\{\mathrm{e}^{-(1+\sqrt{2})z/\sqrt{2}}\cos(3t - \frac{1+\sqrt{2}}{\sqrt{2}}z) \\ & - \mathrm{e}^{-z/\sqrt{2}}\cos(3t - \frac{z}{\sqrt{2}})\}) + \frac{1}{80}(\{11\cos(t - \frac{2}{\sqrt{2}} + \frac{\pi}{2}) \\ & - 2\cos(t - \frac{z}{\sqrt{2}})\}\,\mathrm{e}^{-3z/\sqrt{2}} + 5\{\cos(t - \frac{\sqrt{2}-1}{\sqrt{2}}z) \\ & - \sqrt{2}\cos(t - \frac{\sqrt{2}-1}{\sqrt{2}}z + \frac{\pi}{2}\}\,\mathrm{e}^{-\frac{\sqrt{2}+1}{\sqrt{2}}z} \\ & - \{(10z^2 + 5\sqrt{2}z + 3)\cos(t - \frac{z}{\sqrt{2}}) + (16 - 10\sqrt{2} \end{aligned}$$

$$- 15\sqrt{2}z)\cos(t - \frac{z}{\sqrt{2}} + \frac{\pi}{2})$$
$$- 5(1 - \sqrt{2})\cos(t + \frac{z}{\sqrt{2}} + \frac{\pi}{2})\}\, e^{-z/\sqrt{2}})] + o(\varepsilon^{13/5}), \tag{3.19}$$

where the constants a_i and b_i are obtained in terms of the outer solution as

$$a_4 = \frac{1}{2}j_{0\eta\eta}{}^{(0)}, \quad a_7 = -\frac{2}{3}p_0{}^{(0)}, \quad a_8 = \frac{1}{2}h_{4\eta\eta}{}^{(0)},$$
$$b_3 = \frac{1}{2}p_{0\eta\eta}{}^{(0)}, \quad b_7 = \frac{1}{2}p_{4\eta\eta}{}^{(0)}. \tag{3.20}$$

The functions $[h_0(\eta), p_0(\eta)]$ and $[h_4(\eta), p_4(\eta)]$ associated with the flow in the outer region, are given by the differential sets

$$h_{0\eta\eta\eta} - h_0 h_{0\eta\eta} + \frac{1}{2}h_{0\eta}^2 + 4p_0 = 0, \quad \sigma^{-1}p_{0\eta\eta\eta} - h_0 p_{0\eta\eta} + h_{0\eta}p_{0\eta} = 0, \tag{3.21}$$

with

$$h_0(0) = 0 = h_{0\eta}{}^{(0)}, \quad p_{0\eta}{}^{(0)} = \frac{\sigma\gamma}{4}, \quad h_{0\eta}(\infty) = 0 = p_0(\infty), \tag{3.22}$$

and

$$h_{4\eta\eta\eta} - h_0 h_{4\eta\eta} - h_4 h_{0\eta\eta} + h_{0\eta}h_{4\eta} + 4p_4 = 0, \tag{3.23}$$

$$\sigma^{-1}p_{4\eta\eta\eta} - h_0 p_{4\eta\eta} - h_4 p_{0\eta\eta} + h_{0\eta}p_{4\eta} + h_{4\eta}p_{0\eta} = 0, \tag{3.24}$$

with

$$h_4(0) = 0, \quad h_{4\eta}{}^{(0)} = -\frac{1}{2}, \quad p_{4\eta}{}^{(0)} = 0, \quad h_{4\eta}(\infty) = 0 = p_4(\infty). \tag{3.25}$$

Functions $(h_1, p_1), (h_2, p_2)$ and (h_3, p_3) satisfy homogenous equations, and are therefore zero. In order to solve the systems (21)–(22) and (23)–(25), we introduce new variables defined by

$$\xi = (\frac{\sigma\gamma}{4})^{1/5}\eta, \quad h_0(\eta) = (\frac{\sigma\gamma}{4})^{1/5}h^0(\xi), \quad p_0(\eta) = (\frac{\sigma\gamma}{4})^{1/5}p^0(\xi), \tag{3.26a}$$

$$h_4(\eta) = (\frac{\sigma\gamma}{4})^{-1/5}h^4(\xi), \quad p_4(\eta) = (\frac{\sigma\gamma}{4})^{2/5}p^4(\xi), \tag{3.26b}$$

so that functions (h^0, p^0) and (h^4, p^4) are given by

$$h^{0'''} - h^0 h^{0''} + \frac{1}{2}h^{0'2} + 4p^0 = 0, \quad \sigma^{-1}p^{0'''} - h^0 p^{0''} + h^{o'}p^{o'} = 0, \tag{3.27}$$

with

$$h^0(0) = 0 = h^{0'}(0), \quad p^{0'}(0) = 1, \quad h^{0'}(\infty) = 0 = p^0(\infty), \tag{3.28}$$

and

$$h^4 = \frac{1}{2h^{0''}(0)}[-h^{0'} + \frac{1}{5}p^{0''}(0)(h^0 + \xi h^{0'})], \tag{3.29}$$

$$p^4 = \frac{1}{2h^{0''}(0)}[-p^{0'} + \frac{1}{5}p^{0''}(0)(4p^0 + \xi p^{0'})]. \tag{3.30}$$

In the above expressions, the prime denotes differentiation with respect to ξ. Numerical solutions of the set (27)–(28) have been obtained by Rotem [11] and Chawla and Verma [12] for various values of the Prandtl number σ. It is evident from the expressions (17)–(19) that the Stokes layers exist in the form of two skin-waves associated with viscous and thermal diffusion.

We are now in a position to write the components of the coefficients of the skin-friction at the oscillating plane and the coefficient of heat transfer under the influence of the buoyancy force. These are given by

$$\begin{aligned} H_{zz}(0,t) &= \varepsilon^{4/5}(\frac{\sigma\gamma}{4})^{3/5}h^{0''}_{(0)} - \varepsilon[\frac{1}{\sqrt{2}} + \frac{\sqrt{2}-1}{2}(\cos 2t - \sin 2t)] \\ &+ \varepsilon^{8/5}(\frac{\sigma\gamma}{4})^{1/5}[\frac{2p^0(0)}{h^{0''}(0)} + \frac{3}{10}p^{0''}(0)] \\ &+ \varepsilon^2(\frac{\sigma\gamma}{2})\left[1 - \frac{\sqrt{2}(1+(\sqrt{2}-1)\sigma^{1/2})\sin 2t}{2\sigma^{1/2}(\sigma^{1/2}+1)(\sigma^{1/2}+\sqrt{2})}\right] + o(\varepsilon^{13/5}), \end{aligned} \tag{3.31}$$

$$\begin{aligned} G_z(0,t) &= -\frac{1}{\sqrt{2}}(\cos t - \sin t) + \frac{3}{8}\varepsilon^{9/5}(\frac{\sigma\gamma}{4})^{3/5}h^{0''}(0)\sin t \\ &+ \frac{\varepsilon^2}{\sqrt{2}}[0.0102(\cos 3t - \sin 3t) - (0.0598\cos t + 0.2616\sin t)] + o(\varepsilon^{13/5}), \end{aligned} \tag{3.32}$$

$$\begin{aligned} S_z(0,t) &= \frac{\sqrt{2}\sigma}{4} + \frac{\sigma}{4(\sigma^{1/2}+\sqrt{2})}(\cos 2t - \sin 2t) + \frac{\sigma\varepsilon^{3/5}}{8}(\frac{\sigma\gamma}{4})^{1/5}p^{0''}(0) \\ &+ \frac{3\sigma\varepsilon^{7/5}}{40}(\frac{\sigma\gamma}{4})^{-1/5}\frac{[p^{0''}(0)]^2}{h^{0''}(0)} + o(\varepsilon^{9/5}). \end{aligned} \tag{3.33}$$

The axial inflow induced in the far region is given by

$$H(\infty,t) = h^0(\infty)(\frac{\sigma\gamma\varepsilon^2}{4})^{-1/5}[(\frac{\sigma\gamma}{4})^{2/5} + \frac{\varepsilon^{4/5}}{10}\frac{p^{0''}(0)}{h^{0''}(0)} + o(\varepsilon^{7/5})]. \tag{3.34}$$

For various values of $\sigma, h^{0''}(0), h^0(\infty), p^0(0)$ and $p^{0''}(0)$ are tabulated in Chawla and Verma [12].

The torque exerted on a disk of radius R (where R is large as compared to the thickness of the boundary layer) by the fluid is given by $-\frac{\pi R^4}{2}\rho\Omega(\nu\omega)^{1/2}G_z(0,t)$. The magnitude of the torque associated with the fundamentals ($\cos t$ and $\sin t$) is

$$| N |=| N_0 || 1 + 0.115(\sigma\gamma)^{3/5}h^{0''}(0)\varepsilon^{9/5} - 0.101\varepsilon^2 |, \tag{3.35}$$

where $| N_0 |= \frac{\pi R^4}{4}\rho\Omega(\nu\omega)^{1/2}$. This should be compared with the result of Rosenbalt [3], namely

$$| N |=| N_0 || 1 - 0.101\varepsilon^2 | . \tag{3.36}$$

In view of the fact that $h^{0''}(0)$ is negative for all σ (see [12]), we conclude that buoyancy, in conjunction with viscous and thermal diffusion, decreases the torque exerted by the fluid. The second term in (35) dominates the succeeding term as long as $\sigma\gamma = o(1)$. Only when $\sigma\gamma = o(\varepsilon^{1/3})$, the last term is comparable with the second one. For a fixed value of σ (or γ) the torque exerted on the plane surface decreases as γ (or σ) is increased.

The steady part of the radial shearing stress at the plane, however, increases with γ. This follows from the fact that the axial inflow in the far region is increased by the buoyancy-induced circulation. As a result, the thickness of the outer region of the steady radial flow is less than that of the Rosenblat layer. The buoyancy-dominated flow occurs within a layer of thickness $o([4\nu kc/g\beta\Omega^2]^{1/5})$ in comparison to the Rosenblat layer whose thickness is of $o((\nu\omega/\Omega)^{1/2})$. The outer layer is established as a complicated balance between buoyancy, diffusion (viscous and thermal) and convection.

4. Solution with $\gamma = o(\varepsilon^2), \sigma = o(1)$

In view of the fact that the outer variable is defined as $\xi = \varepsilon^{3/5}(\frac{\sigma\gamma}{4})^{1/5}z$ when $\gamma = o(1)$, we infer that the thickness of the outer region increases as the buoyancy parameter is decreased. For $\gamma = o(\varepsilon^2)$, it is natural to assume the outer region to be of $o(\varepsilon^{-1})$ times as thick as the inner Stokes layers. We thus take $\zeta = \varepsilon z$ as the outer variable with

$$\gamma = \gamma_2\varepsilon^2, \quad H(z,t) = h(\zeta,t), \quad P(z,t) = \varepsilon^2 p(\zeta,t), \quad S(z,t) = s(\eta,t), \tag{4.37}$$

where the functions h, p, and s satisfy

$$h_{\zeta t} = \varepsilon^2[h_{\zeta\zeta\zeta} - hh_{\zeta\zeta} + \frac{1}{2}h_\zeta^2 + 4p], \tag{4.38}$$

$$p_{\zeta t} = \varepsilon^2[\sigma^{-1}p_{\zeta\zeta\zeta} - hp_{\zeta\zeta} + h_\zeta p_\zeta] + \frac{\varepsilon^4\gamma_2}{4}h_{\zeta\zeta}^2, \tag{4.39}$$

$$p_\zeta = \gamma_2 s. \tag{4.40}$$

With $\gamma_2 = o(1)$, the inner solution is now given as

$$\begin{aligned} S(z,t) &= \frac{\sigma}{4}(1 - \mathrm{e}^{-\sqrt{2}z}) + \frac{\sigma}{4(\sigma-2)}[\mathrm{e}^{-\sqrt{2}z}\cos(2t - \sqrt{2}z) - \mathrm{e}^{-\sigma^{1/2}z}\cos(2t - \sigma^{1/2}z)] \\ &\quad + \varepsilon b_1 z + o(\varepsilon^2), \end{aligned} \tag{4.41}$$

$$\begin{aligned} H(z,t) &= \varepsilon[\frac{1}{2\sqrt{2}} - \frac{z}{2} - \frac{\mathrm{e}^{-\sqrt{2}z}}{2\sqrt{2}} + \frac{1}{4}(\mathrm{e}^{-\sqrt{2}z}\cos(2t - \sqrt{2}z + \frac{\pi}{4}) \\ &\quad - \sqrt{2}\mathrm{e}^{-z}\cos(2t - z + \frac{\pi}{4}) + (\sqrt{2}-1)\cos(2t + \frac{\pi}{4}))] \\ &\quad + \varepsilon^2 a_2 z^2 + o(\varepsilon^3), \end{aligned} \tag{4.42}$$

$$G(z,t) = \mathrm{e}^{-z/\sqrt{2}}\cos(t - \frac{z}{\sqrt{2}}) + G_1(z,t)\varepsilon^2 + o(\varepsilon^3), \tag{4.43}$$

where $G_1(z,t)$ is same as the coefficient of ε^2 in (19). The constants a_2 and b_1 are give by

$$a_2 = \frac{1}{2}h_{0\zeta\zeta}(0), \quad b_1 = p_{0\zeta\zeta}(0), \tag{4.44}$$

Table 1.

	$\sigma = 0.2$			$\sigma = 1$		
γ_2	a_2	$-b_1$	$-h_0(\infty)$	a_2	$-b_1$	$-h_0(\infty)$
0	0.147	0	0.751	0.147	0	0.751
0.1	0.387	0.001	1.287	0.116	0.010	0.845
0.25	0.442	0.004	1.516	0.072	0.047	0.923
0.5	0.471	0.008	1.724	0.009	0.097	1.015
1	0.480	0.018	1.965	−0.096	0.207	1.133
2	0.452	0.041	2.245	−0.275	0.447	1.274

	$\sigma = 2$		
γ_2	a_2	$-b_1$	$-h_0(\infty)$
0	0.147	0	0.751
0.1	0.102	0.053	0.785
0.25	0.046	0.157	0.828
0.5	−0.019	0.276	0.880
1	−0.170	0.574	0.952
2	−0.469	1.216	1.046

where functions h_0 and p_0 are obtained as the outer solution $[h = \sum_{n=0}^{\infty} \varepsilon^n h_n(\zeta, t)$ etc.] and satisfy

$$h_{0\zeta\zeta\zeta} - h_0 h_{0\zeta\zeta} + \frac{1}{2} h_{0\zeta}^2 + 4p_0 = 0, \quad \sigma^{-1} p_{0\zeta\zeta\zeta} - h_0 p_{0\zeta\zeta} + h_{0\zeta} p_{0\zeta} = 0, \tag{4.45}$$

with

$$h_0(0) = 0, \quad h_{0\zeta}(0) = -\frac{1}{2}, \quad p_{0\zeta} = \frac{\sigma\gamma_2}{4}, \quad h_{0\zeta}(\infty) = 0 = p_0(\infty). \tag{4.46}$$

For $\gamma = O(\varepsilon^2)$, the torque exerted by the fluid on the plane is given, to order ε^2, by the Rosenblat value (36) and is unaffected by buoyancy. The coefficient of the radial shearing stress at, and the heat transfer from the plane are

$$H_{zz}(0, t) = -\varepsilon[\frac{1}{\sqrt{2}} + \frac{\sqrt{2}-1}{2}(\cos 2t - \sin 2t)] + 2a_2\varepsilon^2 + o(\varepsilon^3), \tag{4.47}$$

$$S_z(0, t) = \frac{\sqrt{2}\sigma}{4} + \frac{\sigma}{4(\sigma^{1/2} + \sqrt{2})}(\cos 2t - \sin 2t) + \varepsilon b_1 + o(\varepsilon^2). \tag{4.48}$$

Equations (45)–(46) were solved numerically by the predictor-corrector method. The values of a_2 and b_1 alongwith the inward axial inflow in the far region, namely $-h_0(\infty)$, are tabulated in Table 1 for various values of γ_2 and σ. A close look at (47) and Table 1 reveals that, for $\sigma = 2$, the mean radial flow reverses its direction in the neighbourhood of the bounding plane, for a specific range of values of the buoyancy parameter γ_2. There is, therefore, an axial outflow from the Stokes layers into the outer layers. But the inward axial inflow from the far off region into the outer region increases with γ_2. It is indicative of the double-decker structure of the flow in the outer boundary layers. In the absence of momentum-bouyancy coupling ($\gamma_2 \to 0$) the amplitude of the outer thermal boundary layer tends to become zero. Under these circumstances $p_0 = 0$ in (45) and we are left with the buoyancy free nonlinear boundary-value problem whose solution has been obtained by Benney [4] and Chawla and Verma [5].

5. Concluding remarks

It is evident from equation (8) that S is $O(1)$. The buoyancy parameter

$$\gamma = \frac{g\beta}{c}(\nu/\Omega)^{1/2}(w/\Omega)^{1/2} = K\varepsilon^{-1/2} \quad \text{(say)}. \tag{5.49}$$

We note that γ contains $\varepsilon^{-1/2}$ as a factor, with $\varepsilon \ll 1$. For $\Omega = 10^{-4}$ rads/sec.

$$\begin{aligned} K &\backsim O(10^{-5}) \quad \text{(atmospheric air)} \\ &\backsim O(10^{-4}) \quad \text{(glycerine and engine oil)} \\ &\backsim O(10^{-3}) \quad \text{(lubricating oils)} \end{aligned}$$

In exotic situations $K > 10^{-3}$. Suitable values of the frequency parameter ε^{-1} (large) make the buoyancy parameter γ truly significant.

There are two important physical aspects of the problem under consideration. Firstly, the torque $| N |$ associated with the fundamentals in equation (35); the buoyancy changes the Rosenblat value appreciably as long as $K \geqslant O(\varepsilon^{5/6}\sigma^{-1})$ with $\varepsilon \ll 1$. Secondly, the axial inflow in the far field (see equation (34) and table 1) is considerably affected for all positive values of K, however small.

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